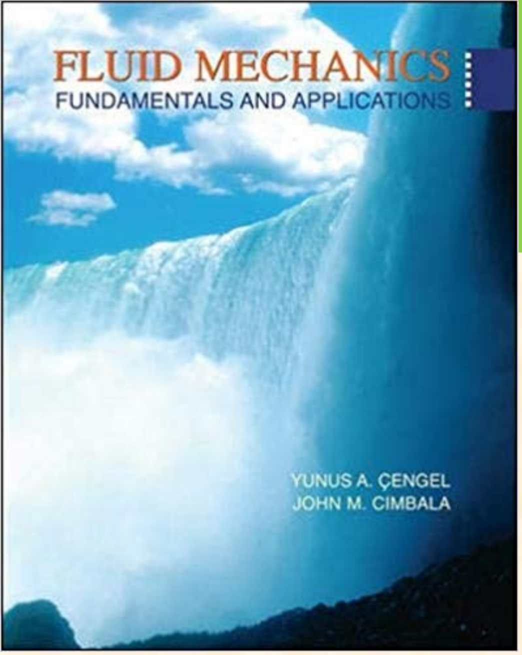


YUNUS A. ÇENGEL and JOHN M. CIMBALA, "Fluid Mechanics: Fundamentals and Applications",1st ed., McGraw-Hill,2006.

Course name

Incompressible Fluid Mechanics

Lecture-01- Chapter-04

Dimensional Analysis And Dynamic Similarity

Lecture slides by

Professor Dr. Thamer Khalif Salem

University of Tikrit

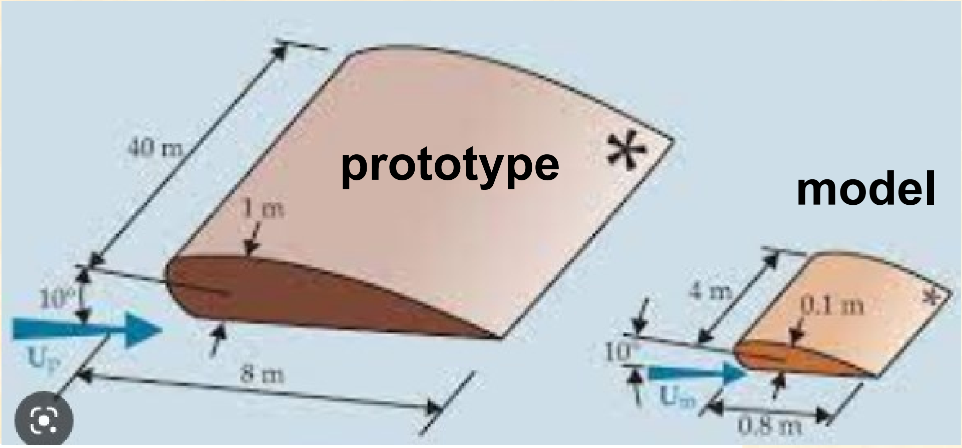
Outline

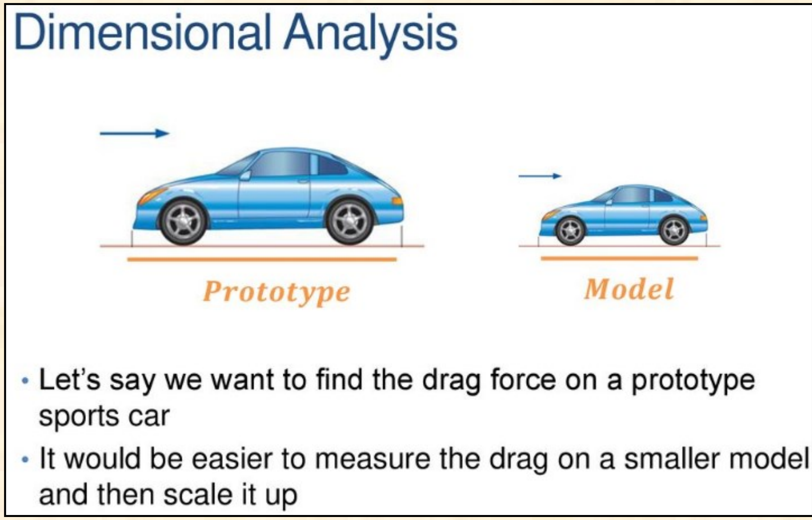
- ***Fluid Basics: Introduction***
- ***Dimensions and Units***
- ***Buckingham π- Theorem or PI Theorem***
  - ✓ **Solution Procedure steps**
  - ✓ ***Explanatory question***
- ***Nondimensional Parameters***
- ***Examples***
- ***Homeworks***

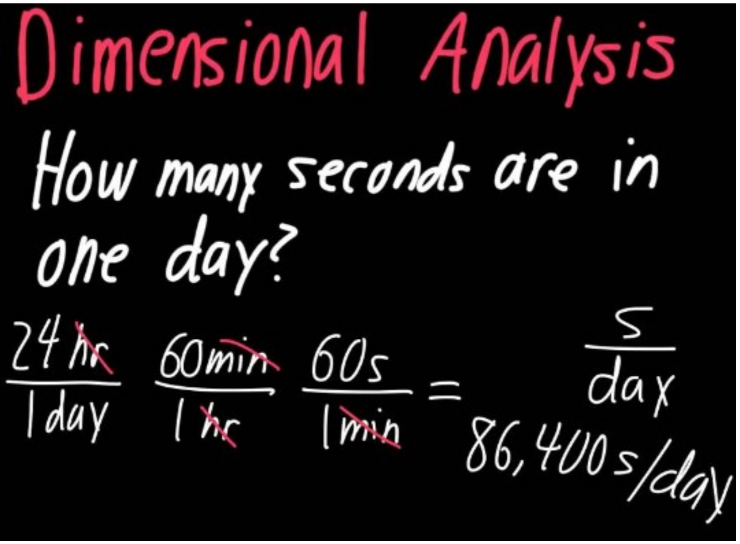
Fluid Basics: Introduction

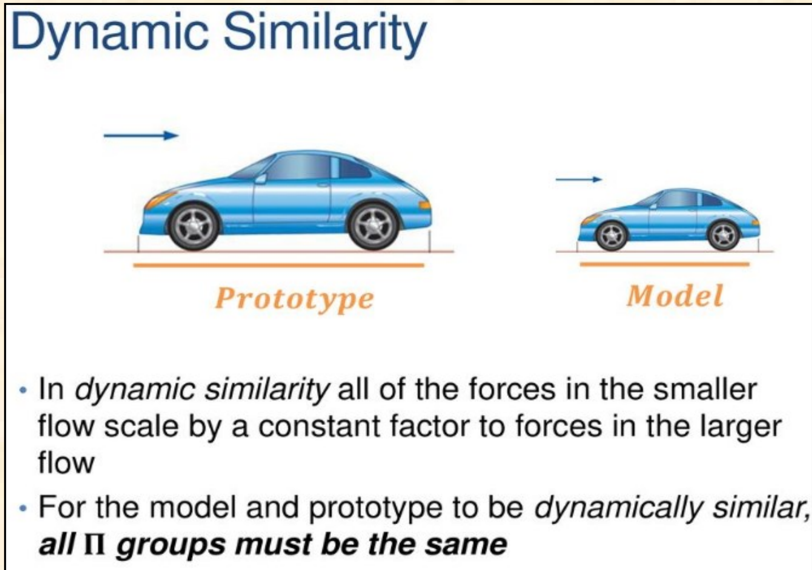
The main principles of **chapter 4** (**chapter 7 in the fluid book**) are:

- Review the concepts of dimensions and units.
- Identify dimensionless groups PI-Theorem. **(Buckingham)**
- Discuss the concept of similarity between a model and a prototype.









- https://www.youtube.com/watch?v=d_WfCwJW0Og
- https://slideplayer.com/slide/16883552/

1- Dimensions and Units

https://www.slideshare.net/YusriYusup/1-dimensions-and-units

DIMENSIONS AND UNITS

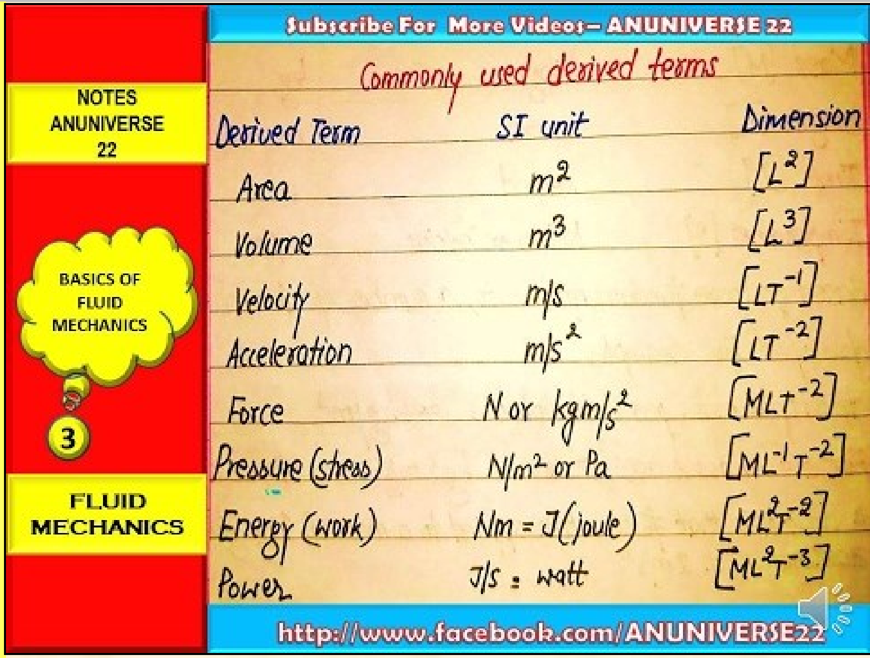
Definition:

Dimensions are basic concepts of physical measurements such as:

- Length = [L]
- Time = [T]
- Mass = [M]
- Temperature = [θ]

Units are terms that precede and describe the dimensions.

1



Derived Term	SI unit	Dimension
Area	m^2	$[L^2]$
Volume	m^3	$[L^3]$
Velocity	m/s	$[LT^{-1}]$
Acceleration	m/s^2	$[LT^{-2}]$
Force	N or kgm/s^2	$[MLT^{-2}]$
Pressure (stress)	N/m^2 or Pa	$[ML^{-1}T^{-2}]$
Energy (work)	Nm = J (joule)	$[ML^2T^{-2}]$
Power	J/s = watt	$[ML^2T^{-3}]$

https://www.youtube.com/watch?v=MLXVIstNQyE

https://analytics.rowsandall.com/2017/12/02/lets-discuss-rowing-metrics/

The dimensions of mechanics are force, length, mass and Time: They are related to Newtons 2^{nd} law of motion

$F = m \cdot a \Rightarrow F = MLT^{-2} \Rightarrow$ نظام أبعاد المعادلات
من دون أشكال ذي متغير
عند الحركات

where

m: mass (kg)

a: acceleration m/s^2

F: Force (N)

M: mass

L: Length

T: Time

N: force

Physical quantity measured	Base unit	SI abbreviation
	mole	mol
	meter	m
	kilogram	kg
	second	s
	kelvin	K
	ampere	A
	candela	cd

www.sciencewithme.com

2- Buckingham π- Theorem or PI Theorem

There are several methods to learn how to generate the non-dimensional parameters, that have been developed for this purpose, but the most popular (and simplest) method is the **method of repeating variables** or **PI-Theorem**, popularized by Edgar Buckingham (1867–1940).

- As a simple example, consider a ball falling in a vacuum as discussed in ***Section 7–23***.

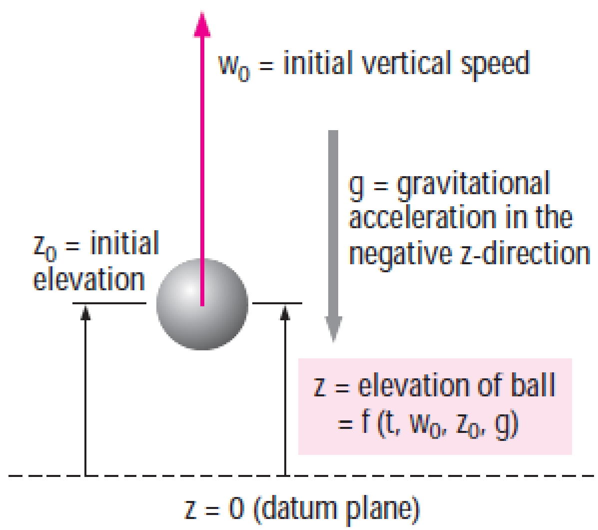


FIGURE 7–23
Setup for dimensional analysis of a ball falling in a vacuum. Elevation z is a function of time t, initial vertical speed w_0, initial elevation z_0, and gravitational constant g.

- **Basic Dimensions of Common Parameters that are used in π- Theorem**

Quantity	Basic Dimensions	
	FLT System (US)	MLT System (SI)
Acceleration	LT^{-2}	LT^{-2}
Angular Velocity	T^{-1}	T^{-1}
Area	L^2	L^2
Mass Density	$FL^{-4}T^2$	ML^{-3}
Weight Density	FL^{-3}	$ML^{-2}T^{-2}$
Force (weight)	F	MLT^{-2}
Kinematic Viscosity	L^2T^{-1}	L^2T^{-1}
Length	L	L
Mass	$FL^{-1}T^2$	M
Power	FLT^{-1}	ML^2T^{-3}
Pressure	FL^{-2}	$ML^{-1}T^{-2}$
Surface Tension	FL^{-1}	MT^{-2}
Velocity	LT^{-1}	LT^{-1}
Viscosity	$FL^{-2}T$	$ML^{-1}T^{-1}$
Volume	L^3	L^3
Volume Flowrate	L^3T^{-1}	L^3T^{-1}
Work, Energy	FL	ML^2T^{-2}

https://www.ecourses.ou.edu/cgi-bin/eBook.cgi?doc=&topic=fl&chap_sec=06.1&page=theory

2- π- Theorem or PI Theorem: Solution Procedure steps

The Buckingham π-Theorem proves That, in a physical problem including هو مجموعة متغيرات n quantities in which there are m dimensions. The quantities can be arranged into $n-m$ groubs -

m: عدد الأبعاد وبصورة عامة يكون عددها ٣

n: - يجب ان لا يقل عن ٤

لإيجاد العلاقة المطلوبة من خلال المتغيرات نتبع مايلي

١- يتم أختيار المتغيرات من خلال منطوق السؤال

٢- تكتب المتغيرات بصيغة الدالة كما يلي $f(V, D, \rho, \mu, g, H \ldots) = 0$

٣- يتم أختيار المتغيرات المتكررة من المتغيرات أعلاه وفق مايلي :-

أ- يجب أن تحتوي الأبعاد MLT فيفضل ثلاثة ρ, V, D

ب- عدم أختيار المتغير المطلوب نسبة الى الأخرين من ضمنها

جـ- أن لا يكون أحدها مشتقة الأخر (السرعة والتعجيل)

د- يفضل أختيار المتغيرات ذات الأبعاد الثنائية أو المفردة

هـ- اذا كان هناك أكثر من متغير ذو أبعاد متشابهة نختار أحدها ضمن المتكررة.

٤- عدد المعادلات (πs) تساوي عدد المتغيرات مطروحاً منها عدد الأبعاد المتكررة

عدد الأبعاد = m ، عدد المتغيرات = n

$n - m$ = No. of πs

Number of dimensionless constants = (Number of variables) – (number of fundamental units involved)

See examples in the table below:

Example:

Given set of variables Constants	*Number of variables Constants*	*Number of fundamental units constants*	*Number of dimensionless*
l, g, t	3	2 (L,T)	3 – 2 = 1
l, v, g	3	2 (L,T)	3 – 2 = 1
p, D, ρ, Q	4	3 (L, M, T)	4 – 3 = 1
F, D, v, ρ, μ	5	3 (L, M, T)	5 – 3 = 2
Q, H, g, γ	4	2 (L,T)	4 – 2 = 2
D, N, μ, p, R	5	3 (L, M, T)	5 – 3 = 2
l, v, ρ, μ, g, R	6	3 (L, M, T)	6 – 3 = 3
$\Delta p, D, l, \rho, \mu, v, t$	7	3 (L, M, T)	7 – 3 = 4
l, v, ρ, μ, E, R	6	3 (L, M, T)	6 – 3 = 3

(iv) The repeating variables, each raised to an index, are grouped with a non-repeating variable to form a dimensionless constant. For example, if the given variables are F, D, v, ρ, g, μ then since there are 6 variables involving 3 fundamental units, we can frame 6 – 3 = 3 dimensionless constants. Since three fundamental units are involved we should select three repeating variables. Obviously, the best choice of repeating variables will be D, v and ρ.

The three dimensionless constants are given by –

$$\pi_1 = D^{a_1} v^{b_1} \rho^{c_1} F, \pi_2 = D^{a_2} v^{b_2} \rho^{c_2} g$$

and

$$\pi_3 = D^{a_3} v^{b_3} \rho^{c_3} \mu.$$

https://www.engineeringenotes.com/fluids/dimensional-analysis/dimensional-analysis-of-a-fluid-methods-equations-buckingham-pi-theorem-and-table/47411

2- π- Theorem or PI Theorem: Solution Procedure steps

٥- تكتب المعادلات كما يلي

$$\pi_1 = V^{x_1} D^{y_1} \rho^{z_1} \mu^{1}$$

$$\pi_2 = V^{x_2} D^{y_2} \rho^{z_2} C^{1}$$

$$\vdots$$

حيث تظاف المتغيرات المتبقيه وترفع للاس (1)
أما المتكررة ترفع للاس (z, y, x)

٦- أيجاد قيم الاس بعد تعويض أبعاد كل متغير

٧- ايجاد كتابة المعادلات بعد تعويض قيم الاس ثم تكتب كافة المعادلات بصيغة الدالة

$$f_1(\pi_1, \pi_2, \pi_3, \ldots) = 0$$

٨- او يكتب المطلوب فمثلاً اذا كان المطلوب ضمن المعادلة الاولى π_1 بالصيغة التالية

$$\pi_1 = f_2(\pi_2, \pi_3, \ldots)$$

Example:

The simple pendulum [edit]

We wish to determine the period *T* of small oscillations in a simple pendulum. It will be assumed that it is a function of the length *L*, the mass *M*, and the acceleration due to gravity on the surface of the Earth *g*, which has dimensions of length divided by time squared. The model is of the form

$$f(T, M, L, g) = 0.$$

(Note that it is written as a relation, not as a function: *T* isn't written here as a function of *M*, *L*, and *g*.)

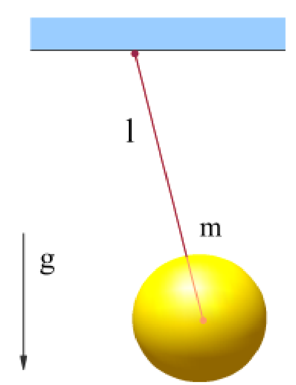
There are 3 fundamental physical dimensions in this equation: time t, mass m, and length ℓ, and 4 dimensional variables, *T*, *M*, *L*, and *g*. Thus we need only 4 − 3 = 1 dimensionless parameter, denoted π, and the model can be re-expressed as

$$f(\pi) = 0,$$

where π is given by

$$\pi = T^{a_1} M^{a_2} L^{a_3} g^{a_4}$$

$$\pi = T^2 M^0 L^{-1} g^1 = gT^2/L.$$



example : [Pando1]

$m = MTL = 3$

$n = MLTg = 4$

$\pi_n = n - m = 4 - 3 = 1 \Rightarrow \pi_1$

$f(\pi_1) = f(MLTg)$

$\pi_1 = M^0 L^0 T^0 = M^{a_1} . L^{a_2} . T^{a_3} . g^1$

$g = \frac{m}{s^2} = LT^{-2}$

$M^0 L^0 T^0 = M^{a_1} . L^{a_2} . T^{a_3} . [LT^{-2}]^1$

mass $\rightarrow 0 = a_1 \Rightarrow a_1 = 0$

Length $\rightarrow 0 = a_2 + 1 \Rightarrow a_2 = -1$

time $\rightarrow 0 = a_3 - 2 \Rightarrow a_3 = 2$

$\therefore \pi_1 = M^0 . L^{-1} . T^2 . g$

$\boxed{\pi_1 = g.T^2/L}$

https://en.wikipedia.org/wiki/Buckingham_%CF%80_theorem

Nondimensional Parameters

Dimensionless Parameters $f_i = r\frac{V^2}{l}$

- Reynolds Number $\quad \mathrm{Re} = \frac{rVl}{m} \qquad f_u = m\frac{V}{l^2}$
- Froude Number $\quad \mathrm{Fr} = \frac{V}{\sqrt{gl}} \qquad f_g = rg$
- Weber Number $\quad W = \frac{V^2 l\rho}{\sigma} \qquad f_s = \frac{s}{l^2}$
- Mach Number $\quad M = \frac{V}{c} \qquad f_{E_v} = \frac{rc^2}{l}$
- Pressure/Drag Coefficients $\quad C_p = \frac{-2(Dp + rgDz)}{rV^2} \qquad C_d = \frac{2\mathrm{Drag}}{\rho V^2 A}$
  - (dependent parameters that we measure experimentally)

Drag coefficient	$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$	$\frac{\text{Drag force}}{\text{Dynamic force}}$
Eckert number	$\mathrm{Ec} = \frac{V^2}{c_p T}$	$\frac{\text{Kinetic energy}}{\text{Enthalpy}}$
Euler number	$\mathrm{Eu} = \frac{\Delta P}{\rho V^2}\left(\text{sometimes } \frac{\Delta P}{\frac{1}{2}\rho V^2}\right)$	$\frac{\text{Pressure difference}}{\text{Dynamic pressure}}$
Fanning friction factor	$C_f = \frac{2\tau_w}{\rho V^2}$	$\frac{\text{Wall friction force}}{\text{Inertial force}}$
Fourier number	$\mathrm{Fo}\ (\text{sometimes } \tau) = \frac{\alpha t}{L^2}$	$\frac{\text{Physical time}}{\text{Thermal diffusion time}}$
Froude number	$\mathrm{Fr} = \frac{V}{\sqrt{gL}}\left(\text{sometimes } \frac{V^2}{gL}\right)$	$\frac{\text{Inertial force}}{\text{Gravitational force}}$
Grashof number	$\mathrm{Gr} = \frac{g\beta\lvert\Delta T\rvert L^3\rho^2}{\mu^2}$	$\frac{\text{Buoyancy force}}{\text{Viscous force}}$
Jakob number	$\mathrm{Ja} = \frac{c_p(T - T_{sat})}{h_{fg}}$	$\frac{\text{Sensible energy}}{\text{Latent energy}}$
Knudsen number	$\mathrm{Kn} = \frac{\lambda}{L}$	$\frac{\text{Mean free path length}}{\text{Characteristic length}}$

52

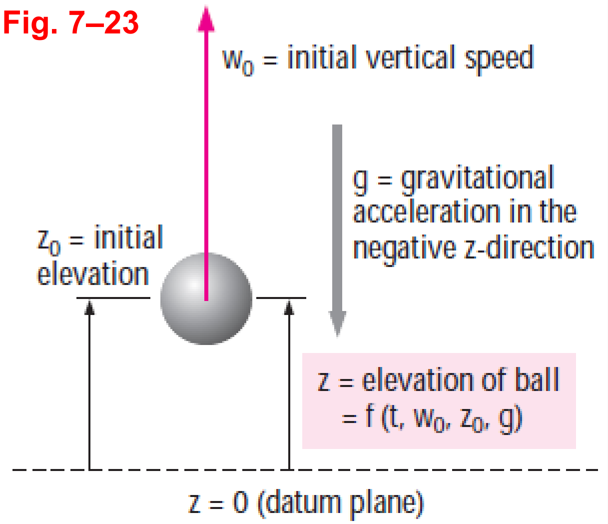
- https://www.slideserve.com/joy/dimensional-analysis-and-similitude
- https://www.slideshare.net/ADDISUDAGNEZEGEYE/fluid-mechanics-chapter-5-dimensional-analysis-and-similitude

Note: Just for briefing, you can See table 7–5 in the fluid mechanics book (pp.287-288) that shows some nondimensional parameters.

2- π- Theorem or PI Theorem: *Explanatory question*

The elevation z of the ball must be a function of time t, initial vertical speed w_0, initial elevation z_0, and gravitational constant g (Fig. 7–23).

Fig. 7–23



Solution: List of relevant parameters: $z = f(t, w_0, z_0, g) \qquad n = 5$

z	t	w_0	z_0	g
$\{L^1\}$	$\{t^1\}$	$\{L^1t^{-1}\}$	$\{L^1\}$	$\{L^1t^{-2}\}$

n=5 & m =2 : π_s= n-m=5-2=3

Repeating parameters: w_0 and z_0

Dependent Π: $$\Pi_1 = z w_0^{a_1} z_0^{b_1} \qquad \text{(7–15)}$$

Dimensions of Π_1: $\{\Pi_1\} = \{L^0t^0\} = \{z w_0^{a_1} z_0^{b_1}\} = \{L^1(L^1t^{-1})^{a_1}L^{b_1}\}$

Time: $\{t^0\} = \{t^{-a_1}\} \qquad 0 = -a_1 \qquad a_1 = 0$

Length: $\{L^0\} = \{L^1L^{a_1}L^{b_1}\} \qquad 0 = 1 + a_1 + b_1 \qquad b_1 = -1 - a_1 \qquad b_1 = -1$

Equation 7–15 thus becomes

$$\Pi_1 = \frac{z}{z_0} \qquad \text{(7–16)}$$

First independent Π: $$\Pi_2 = t w_0^{a_2} z_0^{b_2}$$

Dimensions of Π_2: $\{\Pi_2\} = \{L^0t^0\} = \{t w_0^{a_2} z_0^{b_2}\} = \{t(L^1t^{-1})^{a_2}L^{b_2}\}$

Equating exponents,

Time: $\{t^0\} = \{t^1t^{-a_2}\} \qquad 0 = 1 - a_2 \qquad a_2 = 1$

Length: $\{L^0\} = \{L^{a_2}L^{b_2}\} \qquad 0 = a_2 + b_2 \qquad b_2 = -a_2 \qquad b_2 = -1$

Π_2 is thus

$$\Pi_2 = \frac{w_0 t}{z_0} \qquad \text{(7–17)}$$

2- π- Theorem or PI Theorem: *Explanatory question*

Finally we create the second independent Π (Π_3) by combining the repeating parameters with g and forcing the Π to be dimensionless (Fig. 7–26).

Second independent Π: $\Pi_3 = g w_0^{a_3} z_0^{b_3}$

Dimensions of Π_3: $\{\Pi_3\} = \{L^0 t^0\} = \{g w_0^{a_3} z_0^{b_3}\} = \{L^1 t^{-2} (L^1 t^{-1})^{a_3} L^{b_3}\}$

Equating exponents,

Time: $\{t^0\} = \{t^{-2} t^{-a_3}\}$ $\quad 0 = -2 - a_3 \quad a_3 = -2$

Length: $\{L^0\} = \{L^1 L^{a_3} L^{b_3}\}$ $\quad 0 = 1 + a_3 + b_3 \quad b_3 = -1 - a_3 \quad b_3 = 1$

Π_3 is thus

$$\Pi_3 = \frac{g z_0}{w_0^2} \tag{7–18}$$

Modified Π_3: $$\Pi_{3,\text{ modified}} = \left(\frac{g z_0}{w_0^2}\right)^{-1/2} = \frac{w_0}{\sqrt{g z_0}} = \mathrm{Fr} \tag{7–19}$$

Relationship between Π's: $\Pi_1 = f(\Pi_2, \Pi_3) \quad \rightarrow \quad \dfrac{z}{z_0} = f\left(\dfrac{w_0 t}{z_0}, \dfrac{w_0}{\sqrt{g z_0}}\right)$

Or, in terms of the nondimensional variables z^* and t^* defined previously by Eq. 7–6 and the definition of the Froude number,

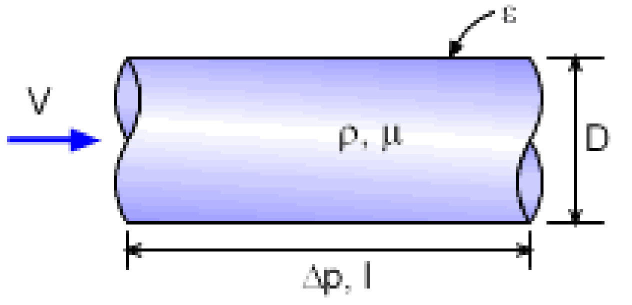
Final result of dimensional analysis: $$z^* = f(t^*, \mathrm{Fr}) \tag{7–20}$$

Nondimensionalized variables: $$z^* = \frac{z}{z_0} \qquad t^* = \frac{w_0 t}{z_0} \tag{7–6}$$

2- π- Theorem or PI Theorem: *Explanatory question*

Example: Flow in a Circular Pipe For a better illustration of the use of dimensional analysis, take fluid flow in a circular pipe.

Solution: n=7 & m =3 : π_s= n-m=7-3=4



Fluid Flow in a Circular Pipe

Quantity	Symbol	MLT
Pressure Drop	Δp	$ML^{-1}T^{-2}$
Pipe Length	l	L
Pipe Diameter	D	L
Fluid Velocity	V	LT^{-1}
Fluid Density	ρ	ML^{-3}
Fluid Viscosity	μ	$ML^{-1}T^{-1}$
Pipe Surface Roughness	ε	L

https://www.ecourses.ou.edu/cgi-bin/eBook.cgi?doc=&topic=fl&chap_sec=06.1&page=theory

The pi terms are then given by:

$$\Pi_1 = \Delta p D^{a1} V^{b1} \rho^{c1}$$
$$\Pi_2 = l D^{a2} V^{b2} \rho^{c2}$$
$$\Pi_3 = \mu D^{a3} V^{b3} \rho^{c3}$$
$$\Pi_4 = \varepsilon D^{a4} V^{b4} \rho^{c4}$$

The exponents of the first pi terms are determined as follows:

$$\Pi_1 = \Delta p D^{a1} V^{b1} \rho^{c1} = (ML^{-1}T^{-2})(L)^{a1}(LT^{-1})^{b1}(ML^{-3})^{c1}$$
$$= M^{(1 + c1)}\, L^{(-1 + a1 + b1 - 3c1)}\, T^{(-2 - b1)}$$

In order for Π_1 to be dimensionless:

M: $1 + c1 = 0$
$c1 = -1$

T: $-2 - b1 = 0$
$b1 = -2$

L: $-1 + a1 + b1 - 3c1 = 0$
$a1 = 3(-1) - (-2) + 1 = 0$

Hence, Π_1 is determined to be $\Delta p/\rho V^2$.

2- π- Theorem or PI Theorem: *Explanatory question*

Since the basic dimension for the pipe length l is L, by inspection, the second pi term is given by (a2 = -1, b2 = 0 and c2 = 0):

$$\Pi_2 = l/D$$

Similarly, the last pi term is given by (a4 = -1, b4 = 0 and c4 = 0):

$$\Pi_4 = \varepsilon/D$$

The exponents of the third pi terms are determined as follows:

$$\Pi_3 = \mu D^{a3} V^{b3} \rho^{c3} = (ML^{-1}T^{-1})(L)^{a3}(LT^{-1})^{b3}(ML^{-3})^{c3}$$
$$= M^{(1 + c3)}\, L^{(-1 + a3 + b3 - 3c3)}\, T^{(-1 - b3)}$$

In order for Π_3 to be dimensionless:

M: $1 + c3 = 0$
$c3 = -1$

T: $-1 - b3 = 0$
$b3 = -1$

L: $-1 + a3 + b3 - 3c3 = 0$
$a3 = 3(-1) - (-1) + 1 = -1$

Hence, Π_3 is determined to be $\mu/\rho DV$. Recognizing that the inverse of the pi term is also dimensionless, the third pi term can also be written as $\rho DV/\mu$, which is the Reynolds number (Re).

For flow in a circular pipe, the pressure drop is then given by

$$\Delta p/\rho V^2 = \text{function}(l/D, \varepsilon/D, Re)$$

https://www.ecourses.ou.edu/cgi-bin/eBook.cgi?doc=&topic=fl&chap_sec=06.1&page=theory

Examples

Example$_1$: Primary Dimensions of Surface Tension

An engineer is studying how some insects are able to walk on water (Fig.7–2). A fluid property of importance in this problem is surface tension (σ_s), which has dimensions of force per unit length. Write the dimensions of surface tension in terms of primary dimensions.

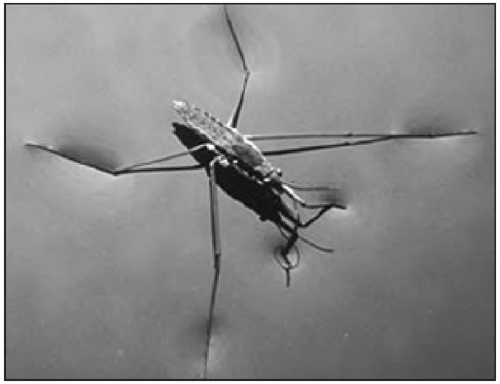


FIGURE 7–2
The water strider is an insect that can walk on water due to surface tension.

SOLUTION The primary dimensions of surface tension are to be determined.

Analysis From Eq. 7–1, force has dimensions of mass times acceleration, or $\{mL/t^2\}$. Thus,

$$\text{Dimensions of surface tension:}\quad \{\sigma_s\} = \left\{\frac{\text{Force}}{\text{Length}}\right\} = \left\{\frac{m \cdot L/t^2}{L}\right\} = \{m/t^2\} \qquad (1)$$

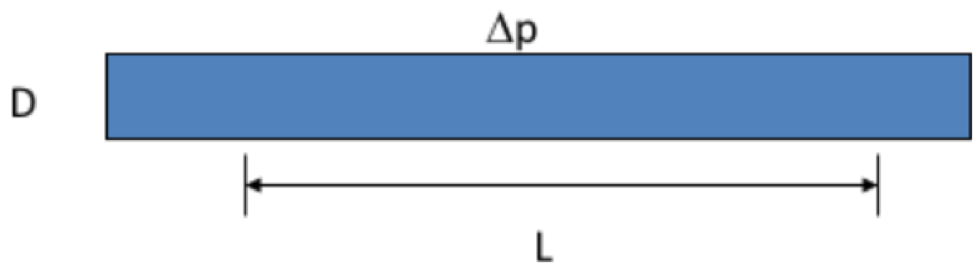
Where:

$$\text{Dimensions of force:}\quad \{\text{Force}\} = \left\{\text{Mass}\,\frac{\text{Length}}{\text{Time}^2}\right\} = \{mL/t^2\} \qquad (7\text{–}1)$$

Homeworks:

HW1: Using Buckingham Pi theorem, determine the dimensionless P parameters involved in the problem of determining pressure drop along a straight horizontal circular pipe shown below. The relevant flow parameters are:

- Δp pressure drop,
- ρ density,
- V averaged velocity,
- μ viscosity
- L pipe length
- D pipe diameter.



Hint: Use the following product groups:

- Group 1 : ρ, V, D, Δp
- Group 2 : ρ, V, D, μ
- Group 3: ρ, V, D, L

https://www.chegg.com/homework-help/questions-and-answers/using-buckingham-pi-theorem-determine-dimensionless-p-parameters-involved-problem-determin-q23654965

- Note:
  - Solve all five Homeworks and sending me the answering next week on Sunday 21March 2024.
  - Read examples 7-7 & 7-8 in the fluid mechanics book (pp.290-293)
- I hope everything is clear for all students

Good luck